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# Quasi Frobenius Lie algebras construction of N=4 superconformal field theories.

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## Abstract

Manin triples construction of N=4 superconformal field theories is considered. The correspondence between quasi Frobenius finite-dimensional Lie algebras and N=4 Virasoro superalgebras is established.

## Introduction.

In connection with numerous string applications extended superconformal field theories (SCFT's) in two dimensions have became increasingly important over the past few years. It is know that a large class of extended SCFT's is obtainable from Kazama-Suzuki models [1], that is, the supercoset constructions assochiated with compact Kahler homogeneous spaces $G/H$. In [2-3] WZNW models were studed, which allow for extended supersymmetry and conditions were formulated that the Lie group must satisfy so that its WZNW model would have extended supersymmetry. In particular, in [3] a correspondence was established between $N = 2, 4$ SCFT's and finite-dimensional Manin triples. From the point of view accepted in [3] Kazama-Suzuki models are particular cases of Manin triple construction of extended SCFT. Indeed they correspond to Manin triples associated with any simple Lie algebra and its parabolic subalgebra. It is intresting to note that there is a similar construction of $N = 3/2$ SCFT based on finite-dimensional Manin pairs [4]. In this paper the conditions under which $N = 2$ SCFT admit $N = 4$ supersymmetric extensions will be investigated from more general positions than it was done in [2-3]. The paper is arranged as follows. In Section 2 we breafly review the Manin triples construction of $N = 2$ SCFT's. In Section 3 we investigate the conditions under which a $N = 2$ SCFT

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associated with any finite-dimensional Manin triple possess $N = 4$ Virasoro superalgebra of symmetries. We will see that it is possible to construct generators of $N = 4$ Virasoro superalgebra if the isotropic subalgebras of Manin triple are quasi Frobenius Lie algebras. Moreover if they are Frobenius then it is possible to construct generators of two different $N = 4$ Virasoro superalgebras. This case corresponds to the "big" $N = 4$ Virasoro superalgebra constructed in [2] and investigated in [7-9]. In section 4 we give some examples of our construction.

## 2. N=2 SCFT and finite-dimensional Manin triples.

We begin with the definition of Manin triple [5]

DEFINITION 2.1. A Manin triple $(g, g_+, g_-)$ consists of a Lie algebra $g$, with nondegenerate invariant inner product $(,)$ and isotropic Lie subalgebras $g_\pm$ such that $g = g_+ \oplus g_-$ as vector space.

For any finite-dimensional Manin triple let us fix any orthonormal basis $\{E^a, E_a, a = 1, ..., d\}$ in algebra $g$ so that $\{E^a\}$- basis in $g_+$, $\{E_a\}$- basis in $g_-$. The brakets and Jacoby identity of $g$ are given by

$$\begin{aligned} [E^a, E^b] &= f^{ab}_c E^c \\ [E_a, E_b] &= f^c_{ab} E^c \\ [E^a, E_b] &= f^a_{bc} E^c - f^{ac}_b E_c \end{aligned} \tag{2.1}$$

$$\begin{aligned} f^{ab}_d f^{dc}_e + f^{bc}_d f^{da}_e + f^{ca}_d f^{db}_e &= 0 \\ f^d_{ab} f^e_{dc} + f^d_{bc} f^e_{da} + f^d_{ca} f^e_{db} &= 0 \\ f^a_{mc} f^{bm}_d - f^a_{md} f^{bm}_c - f^b_{mc} f^{am}_d + & \\ f^b_{md} f^{am}_c &= f^m_{cd} f^{ab}_m \end{aligned} \tag{2.2}$$

In the following we will be needed in the consequence of (2.2)

$$f_m f^{mb}_a + f^m f^b_{ma} = -f^b_{nm} f^{mn}_a \tag{2.3}$$

, where $f_m = f^a_{ma}$, $f^m = f^{ma}_a$. Denote by $\langle , \rangle$ the Killing form of $g$. It is not difficult to calculate

$$\begin{aligned} \langle E^a, E^b \rangle &= 2 f^{ac}_d f^{bd}_c \\ \langle E_a, E_b \rangle &= 2 f^d_{ac} f^c_{bd} \\ \langle E^a, E_b \rangle &= -f^{cd}_b f^a_{cd} - 2 f^{ac}_d f^d_{bc} \end{aligned} \tag{2.4}$$

Let us denote

$$\begin{aligned} B^b_a &= f_c f^{cb}_a + f^c f^b_{ca} \\ A^b_a &= f^d_{ac} f^{bc}_d \end{aligned} \tag{2.5}$$

Then we will have

$$\langle E^b, E_a \rangle = -B^b_a - 2A^b_a \tag{2.6}$$

Let $J^a(z), J_a(z)$ be the generators of affine Kac-Moody algebra $\hat{g}$, which correspond to the fixed basis $\{E^a, E_a\}$, so that currents $J^a$ generate subalgebra $\hat{g_+}$ and currents $J_a$ generate

subalgebra $\hat{g_-}$. The singular OPE's between these currents is the following

$$\begin{aligned} J^a(z)J^b(w) &= -(z-w)^{-2}\frac{1}{2}\langle E^a, E^b\rangle + (z-w)^{-1}f^{ab}_c J^c(w) + reg \\ J_a(z)J_b(w) &= -(z-w)^{-2}\frac{1}{2}\langle E_a, E_b\rangle + (z-w)^{-1}f^{c}_{ab} J_c(w) + reg \\ J^a(z)J_b(w) &= (z-w)^{-2}\frac{1}{2}(q\delta^a_b - \langle E^a, E_b\rangle) + \\ &(z-w)^{-1}(f^a_{bc}J^c - f^{ac}_b J_c)(w) + reg \end{aligned} \tag{2.7}$$

,where $q = 2(k+v)$, $v = \frac{1}{2d}\sum_i Tr(adE^i adE^i)$ and $E^i = E^a, i = a, E^i = E_a, i = a+d$. Let $\psi^a(z), \psi_a(z)$ be free fermion currents which have singular OPE's with respect to the inner product $(,)$

$$\psi^a(z)\psi_b(w) = (z-w)^{-1}\delta^a_b + reg \tag{2.8}$$

ASSERTION 2.2 [3,4] The currents

$$\begin{aligned} G^+ &= \sqrt{\frac{2}{k+v}}(\psi^a J_a - \frac{1}{2}f^c_{ab}\psi^a\psi^b\psi_c) \\ G^- &= \sqrt{\frac{2}{k+v}}(\psi_a J^a - \frac{1}{2}f^{ab}_c\psi_a\psi_b\psi^c) \\ K &= (\delta^b_a + \frac{B^b_a}{k+v}) : \psi^a\psi_b : + \frac{1}{k+v}(f_c J^c - f^c J_c) \\ 2T &= \frac{1}{k+v} : (J^a J_a + J_a J^a) : + \\ &: (\partial\psi^a\psi_a - \psi^a\partial\psi_a) : \end{aligned} \tag{2.9}$$

satisfy the operator products of the $N=2$ Virasoro superalgebra with central charge

$$c = 3(\frac{D}{2} + \frac{A^a_a}{k+v}) \tag{2.10}$$

DEFINITION 2.3. Let $g$ be the Lie algebra with nondegenerate invariant inner product $(,)$ and $R$- complex structure on vector space $g$ skew- symmetric relative to $(,)$. $R$ is complex stucture on Lie algebra $g$ if $R$ satisfies the modified classical Yang-Baxter equation:

$$[Rx, Ry] - R[Rx, y] - R[x, Ry] = [x, y] \tag{2.11}$$

It is not dificult to establish the correspondence between complex Manin triples and complex structures on Lie algebras [3]. Namely for any complex Manin triple $(g, g_+, g_-)$ there is canonic complex structure on Lie algebra such that subalgebras $g_\pm$ are $\pm\imath$- eigenspaces of its. On the other hand, for any real Lie algebra $g$ with nondegenerate invariant inner product and skew- symmetric complex structure $R$ on this algebra one can consider the complexification $g_C$ of $g$. Let $g_\pm$ be $\pm\imath$- eigenspaces of $R$ in algebra $g_C$, then $(g_C, g_+, g_-)$

be the complex Manin triple. Hence we can use formulas (2.9) to build up generators of $N=2$ Virasoro superalgebra.

In connection with the construction described above it is pertient to note the work [13] where very similar construction was considered.

### 3. Quasi Frobenius Lie algebras and N=4 Virasoro superalgebras.

Now we will try to generalize Manin triples construction of $N=$ SCFT for $N=4$ SCFT. $N=4$ Virasoro superalgebra have the following OPE

$$\begin{aligned}
T(z)T(w) &= (z-w)^{-4}\frac{c}{2}+(z-w)^{-2}2T(w)+(z-w)^{-1}\partial T(w)+reg\\
K^i(z)K^j(w) &= (z-w)^{-2}\frac{c}{12}+(z-w)^{-1}\imath\varepsilon^{ijk}K^k(w)+reg\\
T(z)K^i(w) &= (z-w)^{-2}K^i(w)+(z-w)^{-1}\partial K^i(w)+reg\\
K^i(z)G^a(w) &= -(z-w)^{-1}\frac{1}{2}(\sigma^i)^a_bG^b(w)+reg\\
K^i(z)G_a(w) &= (z-w)^{-1}\frac{1}{2}(\sigma^i)^b_aG_b(w)+reg\\
T(z)G^a(w) &= (z-w)^{-2}\frac{3}{2}G^a(w)+(z-w)^{-1}\partial G^a(w)+reg\\
T(z)G_a(w) &= (z-w)^{-2}\frac{3}{2}G_a(w)+(z-w)^{-1}\partial G_a(w)+reg\\
G^a(z)G_b(w) &= (z-w)^{-3}\frac{2c}{3}\delta^a_b+(z-w)^{-2}4(\sigma^i)^a_bK^i(w)+\\
&\quad (z-w)^{-1}(2\delta^a_bT(w)+2(\sigma^i)^a_b\partial K^i(w))+reg
\end{aligned}\tag{3.1}$$

Let us fix some finite- dimensional Manin triple $(g,g_+,g_-)$. From the formulas (3.1) we can see that currents $G^0, G_0$ generate $N=2$ Virasoro superalgebra. With the arguments of preceding section one can establish the existence of the complex structure $R_1$ on Lie algebra $g$. Let us denote:

$$\begin{aligned}
D_0 &= \frac{1}{\sqrt{2}}(G^0+G_0)\\
D_1 &= \frac{-\imath}{\sqrt{2}}(G^0-G_0)\\
D_2 &= \frac{1}{\sqrt{2}}(G^1+G_1)
\end{aligned}\tag{3.2}$$

We can see from (3.1) that the linear combinations

$$\frac{1}{\sqrt{2}}(D_0\pm\imath D_2)=G^0+G_0\pm\imath(G^1+G_1)\tag{3.3}$$

generate another $N=2$ Virasoro superalgebra. Therefore we establish the existence of the second complex stucture $R_2$ on the Lie algebra $g$. Using (3.1) once more it is not dificult to show that for any real numbers $x$ and $y$, such that $x^2+y^2=1$ the currents

$$\frac{1}{\sqrt{2}}(D_0\pm\imath(xD_1+yD_2))$$

also generate $N=2$ Virasoro superalgebra. Hence with the arguments of preceding section we conclude, that the square of the operator

$$S = xR_1 + yR_2 \tag{3.4}$$

is equal to $-1$. This fact implies

$$R_1R_2 + R_2R_1 = 0 \tag{3.5}$$

Next, we intend to show that the existence of two skew- symmetric mutualy anticommuting complex structures on Lie algebra makes it possible to construct generators of $N=4$ Virasoro superalgebra.

Let $g_\pm$ be the eigenspaces of the complex structure $R_1$ on complex Lie algebra $g$. Let us fix the orthonormal basis (2.1) in $g$. In this basis the second complex structure $R_2$ is given by matrix

$$\begin{aligned} R_2E^a &= (r_{11})^a_bE^b + (r_{12})^{ab}E_b \\ R_2E_a &= (r_{22})^b_aE_b + (r_{21})_{ab}E^b \end{aligned} \tag{3.6}$$

and the equation (3.5) equivalet to

$$r_{11} = r_{22} = 0 \tag{3.7}$$

The skew- symmetric condition for $R_2$ takes the form

$$r^T_{12} = -r_{12},\; r^T_{21} = -r_{21} \tag{3.8}$$

Taking into account (3.7), (3.8) one can rewrite the equation $(R_2)^2 = -1$ in the following form

$$r_{21} = -r^{-1}_{12} \tag{3.9}$$

In the following we will denote $r_{12}$ as $r$. In the basis (2.1) equation (2.11) for the $R_2$ takes the form

$$\begin{aligned} r_{ad}f^d_{cb} + r_{bd}f^d_{ac} + r_{cd}f^d_{ba} &= 0 \\ r^{ad}f^{cb}_d + r^{bd}f^{ac}_d + r^{cd}f^{ba}_d &= 0 \end{aligned} \tag{3.10}$$

,where $r^{ab} = (r^{-1})_{ab}$. That is $r$ is 2-cocycle on algebra $g_+$ and $r^{-1}$ is 2-cocycle on $g_-$. In view of (3.9) they should be nondegenerate.

Define fermionic currents

$$\begin{aligned} G^0_x &= \psi^aJ_a - \frac{1}{2}f^c_{ab}\psi^a\psi^b\psi_c + x^0_a\partial\psi^a \\ G_{0x} &= \psi_aJ^a - \frac{1}{2}f^{ab}_c\psi_a\psi_b\psi^c + x^a_0\partial\psi_a \\ G^1_x &= r_{ba}\psi^aJ^b + \frac{1}{2}r_{am}f^{ab}_cr_{bn}r^{ck}\psi^m\psi^n\psi_k + x^1_a\partial\psi^a \\ G_{1x} &= r^{ba}\psi_aJ_b + \frac{1}{2}r^{am}f^c_{ab}r^{bn}r_{ck}\psi_m\psi_n\psi^k + x^a_1\partial\psi_a \end{aligned} \tag{3.11}$$

,where vectors $x^0, x_0, x^1, x_1$ will be determined later and denote

$$\begin{aligned} G_x^0 &= G^0 + \partial x^0, \quad G_{0x} = G_0 + \partial x_0 \\ G_x^1 &= G^1 + \partial x^1, \quad G_{1x} = G_1 + \partial x_1 \end{aligned} \tag{3.12}$$

It is not difficult to show that the conditions that there no singular terms in OPE's $G_x^0 G_x^0$ , $G_{0x}G_{0x}$, $G_x^1 G_x^1$, $G_{1x}G_{1x}$ are equivalent to the equations

$$\begin{aligned} x_a^0 f_{bc}^a = 0, &\quad x_0^a f_a^{bc} = 0 \\ x_a^1 r^{ad} f_d^{bc} = 0, &\quad x_1^a r_{ad} f_{bc}^d = 0 \end{aligned} \tag{3.13}$$

(here we have used (3.10)), that is vectors $x^0, x_0, rx^1, r^{-1}x_1$ are 1-cocycles on subalgebras $g_\pm$. From now we will imply that (3.13) be satisfied.

LEMMA 3.1.

$$\begin{aligned} G_x^0(z)G_x^1(w) = (z-w)^{-2}[\frac{q}{2}r_{ac} - f_{ac}^m f^n r_{mn} \\ -\frac{1}{2}x_m^1 f_{ac}^m - \frac{1}{2}x_b^0(r_{pa}f_c^{pb} - r_{pc}f_a^{pb})]\psi^a\psi^c + \\ (z-w)^{-1}\frac{1}{2}[\frac{q}{2}r_{ac} - f_{ac}^m f^n r_{mn} - \frac{1}{2}x_m^1 f_{ac}^m]\partial(\psi^a\psi^c) + reg. \end{aligned} \tag{3.14a}$$

$$\begin{aligned} G_x^0(z)G_{1x}(w) = -(z-w)^{-3}2x_a^0 x_1^a \\ -(z-w)^{-2}2[(f_a r^{ab} - \frac{x_1^b}{2})(J_b + f_{bd}^c\psi_c\psi^d) + \frac{x_a^0}{2}r^{ca}(J_c + f_{cb}^d r^{bn} r_{dk}\psi_n\psi^k)] \\ -(z-w)^{-1}(f_a r^{ab} - x_1^b)(J_b + f_{bd}^c)\psi_c\psi^d + reg. \end{aligned} \tag{3.14b}$$

$$\begin{aligned} G_{0x}(z)G_x^1(w) = -(z-w)^{-3}2x_0^a x_a^1 \\ -(z-w)^{-2}2[(f^a r_{ab} - \frac{x_b^1}{2})(J^b + f_c^{bd}\psi^c\psi_d) + \frac{x_0^a}{2}r_{ca}(J^c + f_d^{cb} r_{bn} r^{dk}\psi^n\psi_k)] \\ -(z-w)^{-1}(f^a r_{ab} - x_1^b)(J^b + f_c^{bd}\psi^c\psi_d) + reg. \end{aligned} \tag{3.14c}$$

$$\begin{aligned} G_{0x}(z)G_{1x}(w) = (z-w)^{-2}[\frac{q}{2}r^{ac} - f_m^{ac} f_n r^{mn} \\ -\frac{1}{2}x_1^m f_m^{ac} - \frac{1}{2}x_0^b(r^{pa}f_{pb}^c - r^{pc}f_{pb}^a)]\psi_a\psi_c + \\ (z-w)^{-1}\frac{1}{2}[\frac{q}{2}r^{ac} - f_m^{ac} f_n r^{mn} - \frac{1}{2}x_1^m f_m^{ac}]\partial(\psi_a\psi_c) + reg. \end{aligned} \tag{3.14d}$$

PROOF. Let us calculate the operator product

$$G_x^0(z)G_x^1(w) = G^0(z)G^1(w) + G^0(z)\partial x^1(w) + \partial x^0(z)G^1(w) + \partial x^0(z)\partial x^1(w) \tag{3.15}$$

We start by calculating singular terms of $G^0G^1$

$$
\begin{aligned}
G^0(z)G^1(w) = (z-w)^{-2}[\frac{1}{2}r_{bc}(q\delta^b_a - < E^b, E_a >)\psi^a\psi^c + \\
f^n_{am}f^{bc}_d r_{bn}r_{cq}r^{dm}\psi^a\psi^q] + \\
(z-w)^{-1}[\frac{1}{2}r_{bc}(q\delta^b_a - < E^b, E_a >)\partial\psi^a\psi^c + \\
f^n_{am}f^{bc}_d r_{bn}r_{cq}r^{dm}\partial\psi^a\psi^q] + \\
(z-w)^{-1}\frac{1}{2}[-f^n_{am}f^{bc}_d r_{bn}r_{cq}r^{ds}\psi^a\psi^m\psi^q\psi_s + \\
f^n_{am}f^{bc}_d r_{bp}r_{cq}r^{dm}\psi^a\psi^p\psi^q\psi_n]
\end{aligned}
\tag{3.16}
$$

Let us denote

$$
\begin{aligned}
U = \frac{1}{2}r_{bc}(q\delta^b_a - < E^b, E_a >)\psi^a\psi^c + \\
f^n_{am}f^{bc}_d r_{bn}r_{cq}r^{dm}\psi^a\psi^q] \\
V = \frac{1}{2}r_{bc}(q\delta^b_a - < E^b, E_a >)\partial\psi^a\psi^c + \\
f^n_{am}f^{bc}_d r_{bn}r_{cq}r^{dm}\partial\psi^a\psi^q] \\
W = \frac{1}{2}(-f^n_{am}f^{bc}_d r_{bn}r_{cq}r^{ds}\psi^a\psi^m\psi^q\psi_s + \\
f^n_{am}f^{bc}_d r_{bp}r_{cq}r^{dm}\psi^a\psi^p\psi^q\psi_n)
\end{aligned}
$$

We are going to show that

$$
W = 0 \tag{3.17}
$$

In view of (3.9), (3.10) we have

$$
\begin{aligned}
f^{bc}_d r_{bn}r_{cq}r^{ds} = f^{bs}_q r_{bn} - f^{bs}_n r_{bq} \\
f^d_{bc}r^{bp}r^{cq}r_{dm} = f^q_{bm}r^{bp} - f^n_{bp}r^{bq}
\end{aligned}
\tag{3.18}
$$

Taking this equation into account we can represent $W$ as the following

$$
\begin{aligned}
W = ((f^{it}_k r_{iq} - f^{it}_q r_{ik})f^k_{ap} + \\
(f^{im}_a r_{ip} - f^{im}_q r_{ia})f^k_{mq})\psi^a\psi^p\psi^q\psi_t
\end{aligned}
\tag{3.19}
$$

Here it is pertient to make a comment about (3.17). Let us denote

$$
h^s_{qn} = f^{bs}_q r_{bn} - f^{bs}_n r_{bq} \tag{3.20}
$$

In view of (3.9), (3.10) the constants $h^s_{qn}$ determine another Lie structure on vector space $g_-$. From (3.19) we can see that (3.17) is the condition this new Lie- structure is compatible with the old Lie- structure on $g_-$. Using the first equation from (3.10) one can write

$$
\begin{aligned}
(f^{it}_k r_{iq} - f^{it}_q r_{ik})f^k_{ap} = f^{it}_k f^k_{ap}r_{iq} + \\
f^{it}_q(r_{ak}f^k_{pi} + r_{pk}f^k_{ia})
\end{aligned}
\tag{3.21}
$$

After sbstitution this formula in the left hand side of (3.19) we obtain

$$W = (f^i_{ap} f^{kt}_i r_{kq} + 2 f^k_{pi} f^{it}_a r_{kq} - 2 f^t_{pi} f^{ik}_a r_{kq}) \psi^a \psi^p \psi^q \psi_t \tag{3.22}$$

Now (3.17) follows from the last equation out of (2.2). With help of (3.10) and (2.6) $U$ can be transformed into

$$U = (\frac{q}{2} r_{ac} + \frac{1}{2} B^b_a r_{bc} - \frac{1}{4} f^m_{ac} f^{nb}_m r_{bn}) \psi^a \psi^c \tag{3.23}$$

Using (2.5), (3.10) it is not difficult to obtain

$$B^b_a r_{bc} \psi^a \psi^c = -\frac{1}{2} f^m_{ca} f^{nk}_m r_{nk} \psi^a \psi^c \tag{3.24}$$

From (3.10) we can easily find

$$\begin{aligned} f^{nk}_m r_{nk} &= 2 f^n r_{nm} \\ f^m_{nk} r^{nk} &= 2 f_n r^{nm} \end{aligned} \tag{3.25}$$

Hence, we have

$$U = (\frac{q}{2} r_{ab} - f^m_{ab} f^n r_{mn}) \psi^a \psi^c \tag{3.26}$$

Next, transform V

$$\begin{aligned} V = \frac{1}{2}(\frac{1}{2}(q\delta^b_a + B^b_a + 2A^b_a) r_{bc} + f^n_{am} f^{pq}_s r_{pn} r_{qc} r^{sm}) \partial(\psi^a \psi^c) + \\ \frac{1}{2}(\frac{1}{2} q\delta^b_a + B^b_a + 2A^b_a) r_{bc} + f^n_{am} f^{pq}_s r_{pn} r_{qc} r^{sm})(\partial\psi^a \psi^c - \psi^a \partial\psi^c) \end{aligned} \tag{3.27}$$

From (3.18), (2.5) one can get

$$f^n_{am} f^{ps}_t r_{pn} r_{sc} r^{tm} = -\frac{1}{2} B^b_c r_{ba} - A^b_a r_{bc} \tag{3.28}$$

Therefore the first term from this expression is equal to $\frac{1}{2}\partial U$ and the second term is equal to zero. Hence

$$V = \frac{1}{2} \partial U \tag{3.29}$$

The calculations of $U, V, W$ assembled together give the result

$$\begin{aligned} G^0(z) G^1(w) = (z-w)^{-2} (\frac{q}{2} r_{ab} - f^m_{ab} f^n r_{mn}) \psi^a \psi^b + \\ (z-w)^{-1} \frac{1}{2} (\frac{q}{2} r_{ab} - f^m_{ab} f^n r_{mn}) \partial(\psi^a \psi^b) \end{aligned} \tag{3.30}$$

Taking into account the singular terms of $G^0 \partial x^1$ and $\partial x^0 G^1$ the operator product (3.15) is given by

$$\begin{aligned} G^0_x(z) G^1_x(w) = (z-w)^{-2} (\frac{q}{2} r_{ac} - f^m_{ac} f^n r_{mn} - \frac{1}{2} x^1_m f^m_{ac} - \frac{1}{2} x^0_m (r_{na} f^{nm}_c - r_{nc} f^{nm}_a)) \psi^a \psi^c \\ + (z-w)^{-1} \frac{1}{2} (\frac{q}{2} r_{ac} - f^m_{ac} f^n r_{mn} - x^1_m f^m_{ac}) \partial(\psi^a \psi^c) \end{aligned} \tag{3.31}$$

Next, we calculate

$$G_x^0(z)G_{1x}(w) = G^0(z)G_1(w) + G^0(z)\partial x_1(w) + \partial x^0(z)G_1(w) + \partial x^0(z)\partial x_1(w) \tag{3.32}$$

Here we start by calculating singular terms of $G^0 G_1$

$$\begin{aligned} G^0(z)G_1(w) = -(z-w)^{-3}\frac{1}{2}f^n_{am}f^d_{bc}r^{bm}r^{ca}r_{dn} + \\ (z-w)^{-2}[(r^{ba}f^c_{ab}J_c - \frac{1}{2}r^{bc} < E_a, E_b > \psi^a\psi_c) \\ -\frac{1}{2}f^n_{am}f^d_{bc}r^{bm}r^{ca}r_{ds}\psi_n\psi^s + f^n_{am}f^d_{bc}r^{bm}r^{cq}r_{dn}\psi_q\psi^a] + \\ (z-w)^{-1}[(r^{ba}f^c_{ab}\partial J_c - \frac{1}{2}r^{bc} < E_a, E_b > \partial\psi^a\psi_c) - \\ \frac{1}{2}f^n_{am}f^d_{bc}r^{bm}r^{ca}r_{ds}\partial\psi_n\psi^s + f^n_{am}f^d_{bc}r^{bm}r^{cq}r_{dn}\psi_q\partial\psi^a] + \\ (z-w)^{-1}[f^p_{as}f^d_{bc}r_{bs}r^{cq}r_{dm} - \frac{1}{4}f^n_{am}f^d_{bc}r_{bp}r^{cq}r_{dn}]\psi^a\psi^m\psi_p\psi_q \end{aligned} \tag{3.33}$$

Let us denote

$$\begin{aligned} P = -\frac{1}{2}f^n_{am}f^d_{bc}r^{bm}r^{ca}r_{dn} \\ Q = (r^{ba}f^c_{ab}J_c - \frac{1}{2}r^{bc} < E_a, E_b > \psi^a\psi_c) \\ -\frac{1}{2}f^n_{am}f^d_{bc}r^{bm}r^{ca}r_{ds}\psi_n\psi^s + f^n_{am}f^d_{bc}r^{bm}r^{cq}r_{dn}\psi_q\psi^a \\ R = (r^{ba}f^c_{ab}\partial J_c - \frac{1}{2}r^{bc} < E_a, E_b > \partial\psi^a\psi_c) - \\ \frac{1}{2}f^n_{am}f^d_{bc}r^{bm}r^{ca}r_{ds}\partial\psi_n\psi^s + f^n_{am}f^d_{bc}r^{bm}r^{cq}r_{dn}\psi_q\partial\psi^a \\ S = (f^p_{as}f^d_{bc}r_{bs}r^{cq}r_{dm} - \frac{1}{4}f^n_{am}f^d_{bc}r_{bp}r^{cq}r_{dn})\psi^a\psi^m\psi_p\psi_q \end{aligned}$$

First we prove

$$S = 0 \tag{3.34}$$

Let us denote

$$(r * f_-)^{pq}_n = f^d_{bc}r^{bp}r^{cq}r_{dn} \tag{3.35}$$

and rewrite $S$ as follows

$$\begin{aligned} \frac{1}{4}[f^p_{as}(r * f_-)^{sq}_m - f^p_{ms}(r * f_-)^{sq}_a - \\ f^q_{as}(r * f_-)^{sp}_m + f^q_{ms}(r * f_-)^{sp}_a - f^n_{am}(r * f_-)^{pq}_n]\psi^a\psi^m\psi_p\psi_q \end{aligned} \tag{3.36}$$

The right hand side of (3.36) is coboundary of 1-cochain $r * f_-$ with coefficients in $\wedge^2 g_-$. From the other hand the cochain $r * f_-$ is coboundary of 0- cochain $r$

$$(r * f_-)^{pq}_n = r^{bp}f^q_{bn} - r^{bq}f^p_{bn} \tag{3.37}$$

Therefore equation (3.34) is the consequence of nilpotency condition for the coboundary operator of Lie algebra $g_-$. Because $r*f_-$ is coboundary of $r$ it defines bialgebra- structure on $g_-$ [5]. Therefore we can do the change

$$f_d^{bm} \rightarrow (r*f_-)_d^{bm}$$

in the equations (2.2), (2.3). Using these new equations one can get

$$\begin{aligned} -\frac{1}{2} f_{am}^n f_{bc}^d r^{bm} r^{ca} r_{dk} \psi_n \psi^k = \frac{1}{2} f_{am}^n (r*f_-)_k^{ma} \psi_n \psi^k = \\ -\frac{1}{2}(f_m (r*f_-)_k^{mn} + (r*f_-)^m f_{mk}^n) \end{aligned} \tag{3.38}$$

From (3.18), (3,25) one can obtain

$$\begin{aligned} f_m (r*f_-)_s^{mn} &= -\frac{1}{2} r^{ab} f_{ab}^i f_{is}^n \\ (r*f_-)^m f_{ms}^n &= -\frac{1}{2} r^{ab} f_{ab}^m f_{ms}^n \end{aligned} \tag{3.39}$$

Using (3.18) we can get

$$f_{am}^n f_{bc}^d r^{bm} r^{cq} r_{dn} \psi_q \psi^n = -(f_m r^{qm} f_{qk}^n + r^{pn} f_{pq}^m f_{mk}^q) \psi_n \psi^k \tag{3.40}$$

Taking into account (3.38)-(3.40) one can write

$$Q = -2 f_a r^{ab} (J_b + f_{bm}^n \psi_n \psi^m) \tag{3.41}$$

In the same way we rearrangement $R$

$$R = \frac{1}{2} \partial Q \tag{3.42}$$

Finaly for $P$ we obtain

$$P = 0 \tag{3.43}$$

Indeed

$$P = \frac{1}{2} f_{am}^n (r*f_-)_n^{ma} = f_m r^{mn} f_m = 0$$

Summing up the resalts of culculations $P, Q, R, S$ we obtain

$$G^0(z) G_1(w) = -(z-w)^{-2} 2 f_a r^{ac} (J_c + f_{cm}^n \psi_n \psi^m) - (z-w)^{-1} f_a r^{ac} \partial (J_c + f_{cm}^n \psi_n \psi^m) \tag{3.44}$$

After calculation of the singular terms of $G^0 \partial x_1$, $\partial x^0 G_1$, $\partial x^0 \partial x_1$ we will obtain (3.14b)

The calculations of singular terms in operator products $G_{0x} G_{1x}$ and $G_{0x} G_x^1$ are identical with that just we have done.

The proof is completed.

To obtain $N=4$ Virasoro superalgebras operator products one have to put either

$$G_x^0(z)G_{1x}(w) \sim G_{0x}(z)G_x^1(w) \sim 0 \tag{3.45a}$$

either

$$G_{0x}(z)G_{1x}(w) \sim G_x^0(z)G_x^1(w) \sim 0 \tag{3.45b}$$

Therefore there is two possibilities to construct generators of $N=4$ Virasoro superalgebra. We will investigate each possibility.

CASE (3.45a). From (3.14b-c), (3.45a) one can obtain the system of equations

$$\begin{aligned} x_a^0 x_1^a &= 0 \\ x_1^b &= f_a r^{ab} \\ f_a r^{ab}(J_b + f_{bd}^c \psi_c \psi^d) + x_a^0 r^{ca}(J_c + f_{cb}^d r^{bn} r_{dk} \psi_n \psi^k) &= 0 \\ x_b^1 &= f^a r_{ab} \\ f^a r_{ab}(J^b + f_c^{bd} \psi^c \psi_d) + x_0^a r_{ca}(J^c + f_d^{cb} r_{bn} r^{dk} \psi^n \psi_k) &= 0 \end{aligned} \tag{3.46}$$

Its solution is given by

$$\begin{aligned} x_1^b = f_a r^{ab}, \quad x_0^a = f^a \\ x_b^1 = f^a r_{ab}, \quad x_a^0 = f_a \end{aligned} \tag{3.47}$$

Let us substitute the solution (3.47) into (3.14a)

$$\begin{aligned} G_x^0(z)G_x^1(w) = (z-w)^{-2}[\frac{q}{2}r_{ac} - \frac{1}{2}f_{ac}^m f^n r_{mn} - \frac{1}{2}f_m(r_{na}f_c^{nm} - r_{nc}f_a^{nm})]\psi^a\psi^c + \\ (z-w)^{-1}\frac{q}{4}r_{ac}\partial(\psi^a\psi^c) \end{aligned} \tag{3.48}$$

From (2.2), (2.3), (3.18) it follows that

$$f_m(f_a^{mb} r_{bc} - f_c^{mb} r_{ba}) = f_{ac}^b f^m r_{mb} \tag{3.49}$$

Indeed, from (2.2) one can get

$$f_{ac}^k f^m r_{mk} = \frac{1}{2} f_{nm}^k (f_a^{nm} r_{kc} - f_c^{nm} r_{ka}) \tag{3.50}$$

From the other hand, using (2.3), (3.18) one can get

$$f_{nm}^k (f_a^{nm} r_{kc} - f_c^{nm} r_{ka}) = f_m(f_a^{mk} r_{kc} - f_c^{mk} r_{ka}) - f^m f_{ac}^k r_{km} \tag{3.51}$$

Comparing (3.50) and (3.51) we obtain (3.49). Hence one may write

$$G_x^0(z)G_x^1(w) = (z-w)^{-2}\frac{q}{2}r_{ac}\psi^a\psi^c + (z-w)^{-1}\frac{q}{4}r_{ac}\partial(\psi^a\psi^c) \tag{3.52}$$

In the same way we can derive

$$G_{0x}(z)G_{1x}(w) = (z-w)^{-2}\frac{q}{2}r^{ac}\psi_a\psi_c + (z-w)^{-1}\frac{q}{4}r^{ac}\partial(\psi_a\psi_c) \tag{3.53}$$

Motivated by formulas (3.52), (3.53) we redenote currents $G^a_x(z)$, $G_{ax}(z)$, $a=0,1$ by

$$G_{ax} \to \sqrt{\frac{2}{k+v}}G_{ax}, \quad G^a_x \to \sqrt{\frac{2}{k+v}}G^a_x \tag{3.54}$$

and introduce generators of $su(2)$- Kac-Moody algebra

$$\begin{aligned} K^{01} &= \frac{1}{2}r_{ac}\psi^a\psi^c, \\ K_{01} &= \frac{1}{2}r^{ac}\psi_a\psi_c, \\ K &= \psi^a\psi_a \end{aligned} \tag{3.55}$$

Then (3.52), (3.53) shows that

$$\begin{aligned} G^0_x(z)G^1_x(w) &= (z-w)^{-2}4K^{01}(w) + (z-w)^{-1}2\partial K^{01}(w) + reg. \\ G_{0x}(z)G_{1x}(w) &= (z-w)^{-2}4K_{01}(w) + (z-w)^{-1}2\partial K_{01}(w) + reg. \end{aligned} \tag{3.56}$$

As a simple exercise in the application of formulas (3.10), (3.18), (3.24) one may obtain

$$\begin{aligned} K(z)G^0_x(w) &= (z-w)^{-1}G^0_x(w) + reg. \\ K(z)G_{0x}(w) &= -(z-w)^{-1}G_{0x}(w) + reg. \\ K^{01}(z)G^0_x(w) &= 0 \\ K_{01}(z)G_{0x}(w) &= 0 \\ K_{01}(z)G^0_x(w) &= -(z-w)^{-1}G_{1x}(w) + reg. \\ K^{01}(z)G_{0x}(w) &= -(z-w)^{-1}G^1_x(w) + reg. \end{aligned} \tag{3.57}$$

To find the stress-energy tensor $T$ we calculate operator product $G^0_xG_{0x}$, but the result follows from the Manin triple construction for $N=2$ Virasoro superalgebra (2.9)

$$\begin{aligned} T = \frac{1}{2(k+v)}(J^aJ_a + J_aJ^a) + (\partial\psi^a\psi_a - \psi^a\partial\psi_a) + \\ \frac{1}{2(k+v)}\partial(f_aJ^a - f^aJ_a) + \frac{1}{2(k+v)}(f_af^{ab}_c - f^af^b_{ac})\partial(\psi^c\psi_b) \end{aligned} \tag{3.58}$$

Taking into account (3.49) it is not dificult to show that currents (3.55) are dimension one primary fields relative stress-tensor (3.58). It is clear that OPE $G^1_xG_{1x}$ gives us the same stress-energy tensor (3.62) because currents $G^1_x(z), G_{1x}(z)$ can be derived from currents

$G^0_x(z), G_{0x}(z)$ with help of transformation $\psi^a \to r_{ab}\psi^b$, $\psi_a \to r^{ab}\psi_b$. The following lemma sums up our investigation of CASE (3.45a)

LEMMA 3.2. The fermionic currents

$$
\begin{aligned}
G^0 &= \sqrt{\frac{2}{k+v}}(\psi^a J_a - \frac{1}{2} f^c_{ab}\psi^a\psi^b\psi_c + f_a\partial\psi^a) \\
G_0 &= \sqrt{\frac{2}{k+v}}(\psi_a J^a - \frac{1}{2} f^{ab}_c\psi_a\psi_b\psi^c + f^a\partial\psi_a) \\
G^1_x &= \sqrt{\frac{2}{k+v}}(r_{ba}\psi^a J^b + \frac{1}{2} r_{am} f^{ab}_c r_{bn} r^{ck}\psi^m\psi^n\psi_k + f^a r_{ab}\partial\psi^b) \\
G_{1x} &= \sqrt{\frac{2}{k+v}}(r^{ba}\psi_a J_b + \frac{1}{2} r^{am} f^c_{ab} r^{bn} r_{ck}\psi_m\psi_n\psi^k + f_a r^{ab}\partial\psi_b)
\end{aligned}
\tag{3.59}
$$

generate $N=4$ Virasoro superalgebra with central charge

$$
c = 3d \tag{3.60}
$$

stress- energy tensor

$$
\begin{aligned}
T = \frac{1}{2(k+v)}(J^a J_a + J_a J^a) + (\partial\psi^a\psi_a - \psi^a\partial\psi_a) + \\
\frac{1}{2(k+v)}\partial(f_a J^a - f^a J_a) + \frac{1}{2(k+v)}(f_a f^{ab}_c - f^a f^b_{ac})\partial(\psi^c\psi_b)
\end{aligned}
\tag{3.61}
$$

and $su(2)$- Kac-Moody currents

$$
\begin{aligned}
K^{01} &= \frac{1}{2} r_{ac}\psi^a\psi^c \\
K_{01} &= \frac{1}{2} r^{ac}\psi_a\psi_c \\
K &= \psi^a\psi_a
\end{aligned}
\tag{3.62}
$$

CASE (3.45b). From (3.14a), (3.14d), (3.45b) we obtain the following equations

$$
\begin{aligned}
\frac{q}{2} r_{ac} - f^m_{ac} f^n r_{nm} - x^1_m f^m_{ac} &= 0 \\
\frac{q}{2} r_{ac} - f^m_{ac} f^n r_{nm} - x^0_m (r_{na} f^{nm}_c - r_{nc} f^{nm}_a) &= 0 \\
\frac{q}{2} r^{ac} - f^{ac}_m f_n r^{nm} - x^m_1 f^{ac}_m &= 0 \\
\frac{q}{2} r^{ac} - f^{ac}_m f_n r^{nm} - x^m_0 (r^{na} f^c_{nm} - r^{nc} f^a_{nm}) &= 0
\end{aligned}
\tag{3.63}
$$

The first and third equations of this system have the solutions if nondegenerate 2-cocycles $r, r^{-1}$ are the coboundary cocycles:

$$
r_{ac} = r_m f^m_{ac}, \quad r^{ac} = r^m f^{ac}_m \tag{3.64}
$$

In this case the solutions of the first and third equations are given by

$$
\begin{aligned}
x^1_m &= -r_{mn} f^n + \frac{q}{2} r_m \\
x^m_1 &= -r^{mn} f_n + \frac{q}{2} r^m
\end{aligned}
\tag{3.65}
$$

Solving remain equations we find the conditions such that (3.49b) is satisfied

$$
\begin{aligned}
r_{ac} = r_m f^m_{ac}, &\quad r^{ac} = r^m f^{ac}_m \\
x^1_m = -r_{mn} f^n + \frac{q}{2} r_m, &\quad x^m_1 = -r^{mn} f_n + \frac{q}{2} r^m \\
x^0_m = f_m + \frac{q}{2} r^n r_{nm}, &\quad x^m_0 = f^m + \frac{q}{2} r_n r^{nm}
\end{aligned}
\tag{3.66}
$$

Let us substitute the solutions (3.66) into (3.14b):

$$
\begin{aligned}
G^0_x(z) G_{1x}(w) &= (z-w)^{-2} q r^b (J_b + f^c_{bd} \psi_c \psi^d) + (z-w)^{-1} \frac{q}{2} r^b \partial (J_b + f^c_{bd} \psi_c \psi^d) \\
G_{0x}(z) G^1_x(w) &= (z-w)^{-2} q r_b (J^b + f^{bd}_c \psi^c \psi_d) + (z-w)^{-1} \frac{q}{2} r_b \partial (J^b + f^{bd}_c \psi^c \psi_d)
\end{aligned}
\tag{3.67}
$$

Motivated by these formulas we redenote currents $G^a_x(z)$, $G_{ax}(z)$, $a = 0, 1$ by (3.54) and introduce $su(2)$-Kac-Moody currents:

$$
\begin{aligned}
K^0_1 &= r^a (J_a + f^c_{ab} \psi_c \psi^b) \\
K^1_0 &= r_a (J^a + f^{ab}_c \psi^c \psi_b) \\
K &= (\delta^b_a - r_c r^{cd} f^b_{da} - r^c r_{cd} f^{da}_b) \psi^a \psi_a + r_b r^{ba} J_a - r^b r_{ba} J^a
\end{aligned}
\tag{3.68}
$$

The stress-energy tensor may be obtained in a similar way to the CASE (3.45a):

$$
\begin{aligned}
T = &\frac{1}{2(k+v)} (J^a J_a + J_a J^a) + (\partial \psi^a \psi_a - \psi^a \partial \psi_a) + \\
&\frac{1}{2(k+v)} \partial ((f_a + \frac{q}{2} r^b r_{ba}) J^a - (f^a + \frac{q}{2} r_b r^{ba}) J_a) + \\
&\frac{1}{2(k+v)} ((f_a + \frac{q}{2} r^b r_{ba}) f^{ab}_c - (f^a + \frac{q}{2} r_b r^{ba}) f^b_{ac}) \partial (\psi^c \psi_b)
\end{aligned}
\tag{3.69}
$$

Summing up the investigation of CASE (3.45b) we can get the following

LEMMA 3.3 If nondegenerate 2-cocycles are coboundary

$$
r_{ac} = r_m f^m_{ac}, \quad r^{ac} = r^m f^{ac}_m \tag{3.70}
$$

then fermionic currents

$$
\begin{aligned}
G^0 &= \sqrt{\frac{2}{k+v}} (\psi^a J_a - \frac{1}{2} f^c_{ab} \psi^a \psi^b \psi_c + (f_a + \frac{q}{2} r^b r_{ba}) \partial \psi^a) \\
G_0 &= \sqrt{\frac{2}{k+v}} (\psi_a J^a - \frac{1}{2} f^{ab}_c \psi_a \psi_b \psi^c + (f^a + \frac{q}{2} r_b r^{ba}) \partial \psi_a) \\
G^1_x &= \sqrt{\frac{2}{k+v}} (r_{ba} \psi^a J^b + \frac{1}{2} r_{am} f^c_{ab} r_{bn} r^{ck} \psi^m \psi^n \psi_k + (\frac{q}{2} r_a - r_{ab} f^b) \partial \psi^a) \\
G_{1x} &= \sqrt{\frac{2}{k+v}} (r^{ba} \psi_a J_b + \frac{1}{2} r^{am} f^{ab}_c r^{bn} r_{ck} \psi_m \psi_n \psi^k + (\frac{q}{2} r^a - r^{ab} f_b) \partial \psi_a)
\end{aligned}
\tag{3.71}
$$

generate $N=4$ Virasoro superalgebra with central charge

$$c = 3(qr^a r_a - d) \tag{3.72}$$

stress- energy tensor

$$\begin{aligned} T = \frac{1}{2(k+v)}(J^a J_a + J_a J^a) + (\partial\psi^a\psi_a - \psi^a\partial\psi_a) + \\ \frac{1}{2(k+v)}\partial((f_a + \frac{q}{2}r^b r_{ba})J^a - (f^a + \frac{q}{2}r_b r^{ba})J_a) + \\ \frac{1}{2(k+v)}((f_a + \frac{q}{2}r^b r_{ba})f^{ab}_c - (f^a + \frac{q}{2}r_b r^{ba})f^b_{ac})\partial(\psi^c\psi_b) \end{aligned} \tag{3.73}$$

and $su(2)$-Kac-Moody currents

$$\begin{aligned} K^0_1 = r^a(J_a + f^c_{ab}\psi_c\psi^b) \\ K^1_0 = r_a(J^a + f^{ab}_c\psi^c\psi_b) \\ K = (\delta^b_a - r_c r^{cd} f^b_{da} - r^c r_{cd} f^{da}_b)\psi^a\psi_a + r_b r^{ba} J_a - r^b r_{ba} J^a \end{aligned} \tag{3.74}$$

Now we are in a position to formulate the conditions such that N=2 SCFT associated with any finite- dimensional Manin triple possess $N=4$ Virasoro superalgebra of symmetries. To do it let us introduce Drinfeld's definition of quasi Frobenius and Frobenius Lie algebras:

DEFINITION 3.4. [6] Finite- dimensional Lie algebra is called quasi Frobenius Lie algebra if it endowed with nondegenerate 2-cocycle. If its cocycle is coboundary then it is called Frobenius Lie algebra.

Due to this definition we will call quasi Frobenius Manin triple a Manin triple with quasi Frobenius isotropic subalgebras such that the corresponding nondegenerate 2-cocycles are mutualy inverse. If they are coboundary cocycles we will call this Manin triple Frobenius Manin triple.

As a consequence of lemmas 3.2, 3.3 we can get

PROPOSITION 3.5 Any $N=2$ SCFT associated with quasi Frobenius Manin triple admits $N=4$ extention by the formulas (3.59)- (3.62). If a Manin triple is Frobenius Manin triple then $N=2$ SCFT admits two $N=4$ extentions by the formulas (3.59)-(3.62) and (3.70)- (3.74).

Let us make contact with paper [3] where "big" $N=4$ Virasoro superalgebra was constructed. From the formulas of [3] one can observe that the modification (in the notations used in [3])

$$\begin{aligned} T(z) \to \hat{T}(z) = T(z) + (1-\gamma)\partial U(z) \\ G_a(z) \to \hat{G}_a(z) = G_a(z) + 2(1-\gamma)\partial\Gamma_a(z) \end{aligned} \tag{3.75}$$

converts "big" $N=4$ Virasoro superalgebra into usual $N=4$ Virasoro superalgebra with generators (3.70)-(3.74). The modification

$$\begin{aligned} T(z) \to \hat{T}(z) = T(z) + \gamma\partial U(z) \\ G_a(z) \to \hat{G}_a(z) = G_a(z) + 2\gamma\partial\Gamma_a(z) \end{aligned} \tag{3.76}$$

converts "big" $N=4$ Virasoro superalgebra into usual $N=4$ Virasoro superalgebra with generators (3.59)- (3.62). Therefore construction of "big" $N=4$ Virasoro superalgebra is possible only for the Frobenius Manin triple.

## 4. Examples.

EXAMPLE 1. The first example of the Manin triple bases on any simple Lie algebra $g$ with the scalar product $(,)$ and its Cartan decomposition $g = n_- \oplus h \oplus n_+$, $b_+ = h \oplus n_+$, $b_- = h \oplus n_-$. Consider the Lie algebra

$$p = g \oplus \tilde{h} \tag{4.1}$$

where $\tilde{h}$ is the copy of the Cartan subalgebra $h$ and the Lie algebra structure on $p$ is defined by

$$[g, \tilde{h}] = 0 \tag{4.2}$$

On $p$ we define the invariant scalar product

$$< (X_1, H_1), (X_2, H_2) >= (X_1, X_2) - (H_1, H_2) \tag{4.3}$$

If we set

$$\begin{aligned} p_+ &= \{(X, H) \in p | X \in b_+, \quad H = X_h\} \\ p_- &= \{(X, H) \in p | X \in b_-, \quad H = -X_h\} \end{aligned} \tag{4.4}$$

,where $X_h$ is the projection of $X$ on the Cartan subalgebra $h$, then we will have $p = p_+ \oplus p_-$ and $p_+$, $p_-$ are isotropic subalgebras of $p$, wich are isomorphic to Borel subalgebras $b_+$, $b_-$. We will give the explicit $N=4$ Virasoro superalgebra construction in the simplest case

$$g = sl(2, C) \tag{4.5}$$

In this case there is only one way to fix nondegenerate 2- cocycles on isotropic subalgebras $p_\pm$, namely in the orthonormal basis (2.1) they are given by

$$\begin{aligned} r(E_0, E_1) = r_{01} = -r^{-1} \\ r^{-1}(E^0, E^1) = r^{01} = r \end{aligned} \tag{4.6}$$

, where r is arbitrary nonzero complex number. Cocycles (4.6) are coboundary cocycles

$$r_{01} = -r^{-1} f^1_{01}, \quad r^{01} = -r f^{01}_1 \tag{4.7}$$

Therefore formulas (4.1), (4.4)- (4.6) define Frobenius Manin triple. In this case let

$J^a, J_a, \psi^a, \psi_a$, $a = 0, 1$ be the bosonic and fermionic currents with the OPE

$$
\begin{aligned}
J^0(z)J^1(0) &= -z^{-1}J^1(0) + o(z) \\
J_0(z)J_1(0) &= z^{-1}J_1(0) + o(z) \\
J^0(z)J^0(0) &= -z^{-2} + o(z) \\
J_0(z)J_0(0) &= -z^{-2} + o(z) \\
J^0(z)J_1(0) &= z^{-1}J_1(0) + o(z) \\
J^1(z)J_0(0) &= z^{-1}J^1(0) + o(z) \\
J^0(z)J_0(0) &= z^{-2}(k+1) + o(z) \\
J^1(z)J_1(0) &= z^{-2}k - z^{-1}(J_0 + J^0)(0) + o(z) \\
\psi^a(z)\psi_b(0) &= z^{-1}\delta^a_b + o(z)
\end{aligned}
\tag{4.8}
$$

Then the formulas (3.59)- (3.62) have the following form

$$
\begin{aligned}
G^0 &= \sqrt{\frac{2}{k+2}}(\psi^0 J_0 + \psi^1 J_1 - \psi^0\psi^1\psi_1 + \partial\psi^0) \\
G_0 &= \sqrt{\frac{2}{k+2}}(\psi_0 J^0 + \psi_1 J^1 + \psi_0\psi_1\psi^1 - \partial\psi_0) \\
G^1 &= -\sqrt{\frac{2}{k+2}}r^{-1}(\psi^1 J^0 - \psi^0 J^1 + \psi^1\psi^0\psi_0 - \partial\psi^1) \\
G_1 &= \sqrt{\frac{2}{k+2}}r(\psi_1 J_0 - \psi_1 J_0 + \psi_1\psi_0\psi^0 - \partial\psi_1)
\end{aligned}
\tag{4.9}
$$

$$
c = 6 \tag{4.10}
$$

$$
\begin{aligned}
T = \frac{1}{2(k+2)}(J_a J^a + J^a J_a) + \frac{1}{2}(\partial\psi^a\psi_a - \psi^a\partial\psi_a) + \frac{1}{2(k+2)}\partial(J^0 + J_0) \\
K^{01} = -r^{-1}\psi^0\psi^1, \quad K = \psi^a\psi_a, \quad K_{01} = r\psi_0\psi_1
\end{aligned}
\tag{4.11}
$$

For the second $N = 4$ Virasoro superalgebra formulas (3.70)- (3.74) will look like

$$
\begin{aligned}
G^0 &= \sqrt{\frac{2}{k+2}}(\psi^0 J_0 + \psi^1 J_1 - \psi^0\psi^1\psi_1 - (k+1)\partial\psi^0) \\
G_0 &= \sqrt{\frac{2}{k+2}}(\psi_0 J^0 + \psi_1 J^1 + \psi_0\psi_1\psi^1 + (k+1)\partial\psi_0) \\
G^1 &= -\sqrt{\frac{2}{k+2}}r^{-1}(\psi^1 J^0 - \psi^0 J^1 + \psi^1\psi^0\psi_0 + (k+1)\partial\psi^1) \\
G_1 &= \sqrt{\frac{2}{k+2}}r(\psi_1 J_0 - \psi_1 J_0 + \psi_1\psi_0\psi^0 + (k+1)\partial\psi_1)
\end{aligned}
\tag{4.12}
$$

$$
c = 6(k+1) \tag{4.13}
$$

$$
\begin{aligned}
T = \frac{1}{2(k+2)}(J_aJ^a + J^aJ_a) + \frac{1}{2}(\partial\psi^a\psi_a - \psi^a\partial\psi_a) + \frac{(k+1)}{2(k+2)}\partial(-J^0 + J_0) \\
K^0_1 = -r(J_1 - \psi_1\psi^0) \\
K = \psi^0\psi_0 - \psi^1\psi_1 + J_0 + J^0 \\
K^1_0 = -r^{-1}(J^1 + \psi^1\psi_0)
\end{aligned}
\tag{4.14}
$$

This construction was used in [10] to prove N=4 Virasoro superalgebras determinant formula [11-12].

EXAMPLE 2. Let $g$ be simple even- dimensional Lie algebra. In this situation we can represent its Cartan subalgebra $h$ as the direct sum of subspaces isotropic with respect to the Killing form :

$$h = h_+ \oplus h_- \tag{4.15}$$

If we set

$$
\begin{aligned}
p_+ = n_+ \oplus h_+ \\
p_- = n_- \oplus h_-
\end{aligned}
\tag{4.16}
$$

then we will have $g = p_+ \oplus p_-$ and $p_+$, $p_-$ are isotropic subalgebras of $g$. In this example we give the explicit N=4 Virasoro superalgebra construction in the case:

$$g = sl(3, C) \tag{4.17}$$

Let $\{E_3, E_2, E_1, H_1, H_2, E^1, E^2, E^3\}$ be the standard basis in $sl(3, C)$, such that generators $E^1, E^2$ correspond to the simple roots $\alpha_1, \alpha_2$, $E^3$ corresponds to the maximal root $\alpha_3$ and $E_1, ..., E_3$ correspond to the negative roots of $sl(3, C)$. We define invariant inner product $(,)$ on $g$ by the formula

$$(x, y) = Tr(xy) \tag{4.18}$$

, where $x, y \in g$ and are $3\times3$ matrixes. The basises in isotropic subalgebras $p_\pm$ constituting orthonormal basis in $g$ are given by

$$p_+ = \oplus^3_{a=0} CE^a, \quad p_- = \oplus^3_{a=0} CE_a \tag{4.19}$$

, where

$$E^0 = \frac{1}{\sqrt{3}}(H_1 + exp(\imath\frac{\pi}{3})H_2), \quad E_0 = \frac{1}{\sqrt{3}}(H_1 + exp(-\imath\frac{\pi}{3})H_2) \tag{4.20}$$

Next, one need to fix nondegenerate 2-cocycles on isotropic subalgebras. By the direct culculation one can find that the skew- symmetric bilinear form r is cocycle on $p_+$ if the following equations are satisfied

$$r^{12} = \frac{r^{03}}{\alpha_3(E^0)}, \quad r^{13} = r^{23} = 0 \tag{4.21}$$

, where $r^{ab} = r(E^a, E^b)$. Cocycle r is nondegenerate if $r^{03}$ is nonzero. From (4.21) it follows that r is coboundary cocycle

$$
\begin{aligned}
r^{ab} = r^c f^{ab}_c \\
r^a = \frac{r^{0a}}{\alpha_a(E^0)}
\end{aligned}
\tag{4.22}
$$

The same is true for nondegenerate 2- cocycles on subalgebra $p_-$. That is if we put

$$r_a = \frac{1}{\alpha_a(E_0)r^{0a}} \tag{4.23}$$

we obtain nondegenerate coboundary 2- cocycle $r^{-1}$ on $p_-$

$$r_{ab} = r_c f^c_{ab} \tag{4.24}$$

which is invers to the 2- cocycle r on $p_+$. Hence one may conclude the formulas (4.17)-(4.24) define Frobenius Manin triple and (3.63)-(3.66), (3.74)- (3.78) give us two N=4 Virasoro superalgebras.

EXAMPLE 3. We construct N=4 Virasoro superalgebra based on quasi Frobenius Manin triple with 4-dimensional nilpotent isotropic subalgebras. Let $g_+$ be 4- dimensional nondecomposible nilpotent Lie algebra [6]. There is the only one Lie algebra of this type:

$$\begin{gathered} g_+ = \oplus_{a=1}^4 CE^a \\ [E^1, E^2] = E^3, \quad [E^1, E^3] = E^4 \end{gathered} \tag{4.25}$$

(other brakets are equal to zero). By the direct calculations one obtain that the skew-symmetric bilinear form r on $g_+$ is cocycle if

$$r^{24} = r^{34} = 0 \tag{4.26}$$

and cocycle r is nondegenerate if

$$r^{14} \neq 0, r^{23} \neq 0 \tag{4.27}$$

Fathermore any 2-cocycle r is coboundary iff

$$r^{14} = r^{23} = 0 \tag{4.28}$$

From (4.26)- (4.28) it follows that if the equations (4.26), (4.27) are satisfied then $g_+$ will be quasi Frobenius Lie algebra. For simplicity we set

$$r^{12} = r^{13} = 0 \tag{4.29}$$

Then the invers matrix $r^{-1}$ have the following nonzero elements

$$\begin{aligned} r_{14} = -r_{41} = \frac{1}{r^{14}} \\ r_{23} = -r_{32} = \frac{1}{r^{23}} \end{aligned} \tag{4.30}$$

Having $r^{-1} \in \wedge^2 g_+$ one can use it to define the coboundary bialgebra structure on $g_+$ [5]. Let $g_-$ be the dual space to $g_+$ and $E_1, ..., E_4$ be the dual basis to the basis (4.25)

$$g_- = \oplus_{a=1}^4 CE_a, \quad (E_a, E^b) = \delta^b_a \tag{4.31}$$

then the Lie algebra structure on $g_-$ defined by coboundary cocommutator on $g_+$ is given by

$$[E_4, E_2] = \frac{1}{r^{23}} E_1, \quad [E_4, E_1] = -\frac{1}{r^{14}} E_2 \tag{4.32}$$

In view of one- to- one correspondence between Lie bialgebras and Manin triples [5] we obtain the Manin triple $(g, g_+, g_-)$. Moreover $g_-$ is also quasi Frobebius Lie algebra because as it follows from (4.32) $r^{-1}$ defines the isomorphism of Lie algebras

$$r^{-1} : g_- \to g_+$$

such that preimage of the cocycle $r$ is equal to $r^{-1}$. Therefore we conclude that the formulas (4.25)- (4.32) define quasi Frobenius Manin triple, and formulas (3.59)- (3.62) give us N=4 Virasoro superalgebra.

## REFERENCES


[1] Y.Kazama, H.Suzuki, *Mod.Phys.Lett* **A4** (1989) 235; *Phys.Lett.* **216B** (1989) 112; *Nucl.Phys.* **B321** (1989) 232.

[2] P.Spindel, A.Sevrin, W.Troost, A.Van Proeyen, *Nucl.Phys* **B308** (1988) 662; **B311** (1989/89) 465.

[3] S.Parkhomenko, *Zh. Eksp. Teor. Fiz.* **102** (July 1992) 3-7.

[4] E.Getzler, Manin Pairs and topological Field Theory, *MIT-preprint* (??? 1994).

[5] V.G.Drinfeld, Quantum groups, *Proc. Int. Cong. Math., Berkley, Calif.* (1986) 798.

[6] A.G.Elashvili, Frobenius Lie algebras 2, *Works of Tbilissi Math. Institute* **v.LXXVII** (1985) ???.

[7] M.Gunaydin, J.L.Petersen, A.Taormina, A.Van Proeyen, *Nucl.Phys.* **B322** (1989) 402.

[8] J.L.Petersen, A.Taormina, *CERN-TH.5446/89.*; *EFI-90-61* (August 1990); *CERN-TH.5503/89.*

[9] H.Ooguri, J.L.Petersen, A.Taormina, *Nucl. Phys.* **B368** (1992) 611.

[10] S.Matsuda, *Phys.Lett.* **282B** (1992) 56;

[11] W.Boucher,D.Friedan,A.Kent, *Phys.Lett.* **172B** (1986) 316;

[12] V.K.Dobrev, *Phys.Lett.* **186B** (1987) 43;

[13] D.I.Gurevich, V.V.Lychagin, V.N.Rubtsov, Nonholonomic filtration of cohomologies of Lie algebras and "Large brackets", *Translated Matematicheskie Zametki* **Vol.52, No.1** (1992) 36.